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Direct perturbation theory for a nearly integrable nonlinear equation with application to dark-soliton solutions

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Abstract

Since the linearized equation for a nearly integrable nonlinear equation is essentially a linear equation, the direct perturbation theory is now developed in the frame of Green's function theory of linear differential equations. The perturbed NLS equation with normal dispersion is treated as an example. Some obscure points in previous works about this equation are clarified.

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1. Introduction

There are mainly two kinds of perturbation theories for nearly integrable nonlinear equations after the inverse scattering transform (IST) method was proposed. The first one bound with IST was well established and a comprehensive survey has been done by Kivshar and Malomed [1]. The second one is the direct perturbation theory first introduced by Gorschkov and his colleagues [2, 3]. Since then, there has been a large amount of work on this method (see, e.g. [4–8] and references therein), and it remains an open research area up to now [9–15].

The general steps to construct the direct perturbation theory are (1) deriving a linearized equation of the perturbed nonlinear equation; (2) finding the solutions of the linearized equation; (3) introducing the adjoint solutions; (4) deriving orthogonality relation of these two kinds of solutions; (5) constructing and proving the completeness relation of these solutions. Kaup found that the solutions of linearized equation of the nonlinear Schrödinger equation (NLS equation) are squared Jost solutions [16], and similar results were later obtained for other nearly integrable equations. Their orthogonality relations were usually derived from Wronskian with various forms. On the other hand, the completeness relations were commonly derived from the generalized Marchenko equations, which is still the most difficult step for the direct perturbation theory applied in different nonlinear equations (see, [13, 17, 18] for Korteweg–de Vries (KdV), NLS and NLS$^+$ equations, respectively).

However, these strategies do not work well for the derivative NLS equation (DNLS equation) with corrections because its Wronskian does not have usual properties and its Marchenko equation is too complicated [19, 20]. As a result, we have to find another way to derive the orthogonality and prove the completeness of basic solutions for the linearized equation. Through the paper of Mann [9, 10], we noted that Seeger *et al* [21, 22] had investigated the influence of weak perturbations on the kink solution of the sine-Gordon equation. As the linearized equation of nearly integrable nonlinear equation is essentially a linear equation, they tried to deal with it based upon Green's function theory of linear equation. As the IST method was not raised at that time, their results are not satisfactory. But it is a very good idea to employ Green's function theory to deal with the DNLS equation with some corrections [23]. This way, the adjoint operator of the linearized operator is introduced through integration by parts and then its adjoint solutions are obtained. The orthogonality of the basic solutions and its adjoint solutions is derived by (1+1)-dimensional Green's theorem, and then their completeness is directly shown by Green's function theory. The whole procedure is concise, easy to understand and distinct from other methods for formulating the direct perturbation theory.

In this work, we take the NLS equation with non-vanishing boundary (NLS^+ equation) as an example to show the effects of this kind of direct perturbation theory, which can be compared with what we have done previously [12, 13]. Following our approach, some obscure points in previous works [11, 14] on this equation have been clarified, and a detailed discussion is given at the end of this paper. Based upon Green's function theory, it seems that a general and systematic formalism for the direct perturbation theory should be presented.

2. Perturbed NLS^+ equation

The NLS^+ equation with corrections can be written as

$$\mathrm{i}v_t - v_{xx} + 2(|v|^2 - \rho^2)v = \mathrm{i}\epsilon p[v], \tag{1}$$

where ϵ is a small positive parameter and $p[v]$ is a functional of v. The aim of the perturbation method is to obtain the solution of (1) to the first order of ϵ under the initial condition $v(x,0) = u(x,0)$, where $u(x,0)$ is the expression of the dark-soliton solution of the unperturbed NLS^+ equation at $t = 0$.

It is known that the parameters characterizing the soliton shall be changed in the first-order approximation as the corrections present, and it is rational to assume that [8]

$$v = u^{\mathrm{a}} + \epsilon q, \tag{2}$$

where u^{a} is the adiabatic solution (the same functional form as the exact soliton solution with the parameters depending linearly on ϵt yet) and ϵq is the correction term. Substituting (2) into (1), we get

$$\mathrm{i}q_t - q_{xx} + 2(2|u|^2 - \rho^2)q + 2u^2\bar{q} = \mathrm{i}P[u], \tag{3}$$

and $P[u]$ is the effective source,

$$P[u] = p[u] - s[u], \qquad s[u] = \frac{\mathrm{d}}{\mathrm{d}\tau}u, \tag{4}$$

in which $\tau = \epsilon t$ is the slow time in the multi-time expansion theory [24]. Since (3) is of the order of ϵ and u on its left-hand side is an exact soliton solution, we can rewrite (3) and its complex conjugate together in an operator form

$$\mathcal{L}(u)\boldsymbol{q} = \mathrm{i}\boldsymbol{P}, \tag{5}$$

where the linearized operator

$$\mathcal{L}(u) = \begin{pmatrix} \mathrm{i}\partial_t - \partial_{xx} + 2(2|u|^2 - \rho^2) & 2u^2 \\ -2\bar{u}^2 & \mathrm{i}\partial_t + \partial_{xx} - 2(2|u|^2 - \rho^2) \end{pmatrix}, \tag{6}$$

and $\boldsymbol{q} = (q\ \bar{q})^T$, $\boldsymbol{P} = (P\ \bar{P})^T$, with the initial condition $\boldsymbol{q}(x, t = 0) = (0\,0)^T$.

The eigen-solution of the operator $\mathcal{L}(u)$ with zero eigenvalue is called its basic solution. Similarly to the work of Kaup [16], we can find

$$\mathcal{L}(u)W = 0, \qquad W = \begin{pmatrix} w_1^2 \\ w_2^2 \end{pmatrix}, \tag{7}$$

where $w = (w_1\ w_2)^T$ is a full Jost function, i.e., a solution which satisfies both Lax equations.

3. Adjoint linearized operator

We now introduce the adjoint operator through integration by parts. Considering

$$\boldsymbol{g}^T\mathcal{L}\boldsymbol{f} - \boldsymbol{f}^T\mathcal{L}^A\boldsymbol{g} = \text{divergence}, \tag{8}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are arrays with two components, the adjoint operator is obviously

$$\mathcal{L}^A(u) = \begin{pmatrix} -\mathrm{i}\partial_t - \partial_{xx} + 2(2|u|^2 - \rho^2) & -2\bar{u}^2 \\ 2u^2 & -\mathrm{i}\partial_t + \partial_{xx} - 2(2|u|^2 - \rho^2) \end{pmatrix}, \tag{9}$$

and the divergence term is

$$\mathrm{i}\partial_t(f_1g_1 + f_2g_2) - \partial_x(f_{1x}g_1 - f_1g_{1x} - f_{2x}g_2 + f_2g_{2x}). \tag{10}$$

Comparison of (6) and (9) gives

$$-\sigma_2\mathcal{L}(u)\sigma_2 = \mathcal{L}^A(u), \tag{11}$$

that indicates the basic solution of the adjoint operator

$$\mathcal{L}^A(u)W^A = 0, \qquad W^A = -\mathrm{i}\sigma_2 W = \begin{pmatrix} -w_2^2 \\ w_1^2 \end{pmatrix}, \tag{12}$$

where σ_j, $j = 1, 2, 3$ are the Pauli matrices.

4. Choice of independent eigenfunctions

From the inverse scattering transform, the full Jost functions are

$$\phi(x, t, \zeta) = h(t, \zeta)\phi(x, \zeta), \qquad \tilde{\phi}(x, t, \zeta) = h(t, \zeta)^{-1}\tilde{\phi}(x, \zeta), \tag{13}$$

$$\tilde{\psi}(x, t, \zeta) = h(t, \zeta)\tilde{\psi}(x, \zeta), \qquad \psi(x, t, \zeta) = h(t, \zeta)^{-1}\psi(x, \zeta), \tag{14}$$

where

$$(\phi(x, \zeta)\tilde{\phi}(x, \zeta)) \to \mathrm{e}^{\mathrm{i}\frac{1}{2}\alpha\sigma_3}E(x, \zeta), \qquad \text{as} \quad x \to -\infty, \tag{15}$$

$$(\tilde{\psi}(x, \zeta)\psi(x, \zeta)) \to E(x, \zeta), \qquad \text{as} \quad x \to \infty, \tag{16}$$

and

$$h(t, \zeta) = \mathrm{e}^{\mathrm{i}2\kappa\lambda t}, \qquad E(x, \zeta) = \begin{pmatrix} 1 & -\mathrm{i}\rho\zeta^{-1} \\ \mathrm{i}\rho\zeta^{-1} & 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\kappa x\sigma_3}, \tag{17}$$

where $\kappa = \sqrt{\lambda^2 - \rho^2}$ is a double-value function of the original spectrum parameter λ. To avoid the complexity of Riemann surface, an auxiliary parameter ζ is introduced to make

$$\kappa = \tfrac{1}{2}(\zeta - \rho^2\zeta^{-1}), \qquad \lambda = \tfrac{1}{2}(\zeta + \rho^2\zeta^{-1}), \tag{18}$$

single-valued functions of ζ [25].

As two values of ζ correspond to one value of κ or λ, we have to put a restriction to the domain of ζ, for example, $|\zeta| > \rho$. In this restriction, the independent Jost solutions are $\psi(x,\zeta)$ and $\tilde{\psi}(x,\zeta)$ (or equally, $\phi(x,\zeta)$ and $\tilde{\phi}(x,\zeta)$) and the corresponding independent squared Jost solutions are $\Psi(x,\zeta)$ and $\tilde{\Psi}(x,\zeta)$. Then the adjoint squared Jost solutions are $\Phi^A(x,\zeta)$ and $\tilde{\Phi}^A(x,\zeta)$.

5. (1 + 1)-dimensional Green's theorem and orthogonality

Assuming that $\boldsymbol{f} = \Psi(x,t,\zeta)$ in (8) is the basic solution to the linearized operator $\mathcal{L}(u)$, and $\boldsymbol{g} = \Phi^A(x,t,\zeta')$ to the adjoint one $\mathcal{L}^A(u)$, the lhs (left-hand side) of (8) should be equal to zero, and the integration of the divergence term on its rhs (right-hand side) over $(-L, L)$ for x and $(0,t)$ for t yields

$$\mathrm{i}\int_{-L}^{L}\mathrm{d}x\left\{(f_1g_1 + f_2g_2)\big|_{t=0}^{t=t}\right\} = \int_0^t \mathrm{d}t\left\{[f_{1x}g_1 - f_1g_{1x} - f_{2x}g_2 + f_2g_{2x}]\big|_{x=-L}^{x=L}\right\}. \tag{19}$$

The time factor of the integrand on the lhs of this equation is $\mathrm{e}^{-\mathrm{i}4(\kappa\lambda-\kappa'\lambda')t}\big|_{t=0}^{t=t}$, and the one on the rhs is simply $\mathrm{e}^{-\mathrm{i}4(\kappa\lambda-\kappa'\lambda')t}$; after the definite integration with t, it yields $\frac{1}{-\mathrm{i}4(\kappa\lambda-\kappa'\lambda')}\mathrm{e}^{-\mathrm{i}4(\kappa\lambda-\kappa'\lambda')t}\big|_{t=0}^{t=t}$. We eliminate the factor $\mathrm{e}^{-\mathrm{i}4(\kappa\lambda-\kappa'\lambda')t}\big|_{t=0}^{t=t}$ from the two sides and use $\boldsymbol{f}, \boldsymbol{g}$ to denote the time-independent square Jost solutions in the following text; then (19) reduces to

$$\mathrm{i}\int_{-L}^{L}\mathrm{d}x(f_1g_1 + f_2g_2) = \frac{1}{-\mathrm{i}4(\kappa\lambda - \kappa'\lambda')}[f_{1x}g_1 - f_1g_{1x} - f_{2x}g_2 + f_2g_{2x}]\big|_{x=-L}^{x=L}. \tag{20}$$

As $x = L \to \infty$,

$$\boldsymbol{f} = \Psi(x,\zeta) \to \begin{pmatrix} -\rho^2\zeta^{-2} \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}2\kappa x}\Big|_{x=L}, \tag{21}$$

$$\boldsymbol{g} = \Phi^A(x,\zeta') \to a(\zeta')^2 \begin{pmatrix} \rho^2\zeta'^{-2} \\ 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}2\kappa' x}\Big|_{x=L}, \tag{22}$$

the reflection term is absent because the un-perturbed state is pure-soliton state. Similarly, as $x = -L \to -\infty$, we have

$$\boldsymbol{g} = \Phi^A(x,\zeta') \to \mathrm{e}^{-\mathrm{i}\alpha\sigma_3} \begin{pmatrix} \rho^2\zeta'^{-2} \\ 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}2\kappa' x}\Big|_{x=-L}, \tag{23}$$

$$\boldsymbol{f} = \Psi(x,\zeta) \to a(\zeta)^2\,\mathrm{e}^{\mathrm{i}\alpha\sigma_3} \begin{pmatrix} -\rho^2\zeta^{-2} \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}2\kappa x}\Big|_{x=-L}. \tag{24}$$

It is easy to see that $\boldsymbol{f}_x = \mathrm{i}2\kappa\boldsymbol{f}$ and $\boldsymbol{g}_x = -\mathrm{i}2\kappa'\boldsymbol{g}$, therefore, as $L \to \infty$, the numerator on the rhs of (20) equals

$$-\mathrm{i}2(\kappa + \kappa')(1 + \rho^4\zeta^{-2}\zeta'^{-2})[a(\zeta')^2\,\mathrm{e}^{\mathrm{i}2(\kappa-\kappa')L} - a(\zeta)^2\,\mathrm{e}^{-\mathrm{i}2(\kappa-\kappa')L}]; \tag{25}$$

noting

$$2(\kappa + \kappa') = (\zeta + \zeta')(1 - \rho^2\zeta^{-1}\zeta'^{-1}), \tag{26}$$

$$4(\kappa\lambda - \kappa'\lambda') = (\zeta^2 - \zeta'^2)(1 + \rho^4\zeta^{-2}\zeta'^{-2}), \tag{27}$$

equation (20) becomes

$$\int_{-L}^{L} \mathrm{d}x(f_1g_1 + f_2g_2) = \frac{1 - \rho^2\zeta^{-1}\zeta'^{-1}}{\mathrm{i}(\zeta - \zeta')}[a(\zeta')^2\,\mathrm{e}^{\mathrm{i}2(\kappa-\kappa')L} - a(\zeta)^2\,\mathrm{e}^{\mathrm{i}2(\kappa'-\kappa)L}], \tag{28}$$

or

$$\mathrm{i}(\zeta - \zeta')\int_{-L}^{L} \mathrm{d}x(f_1g_1 + f_2g_2) = (1 - \rho^2\zeta^{-1}\zeta'^{-1})[a(\zeta')^2\,\mathrm{e}^{\mathrm{i}2(\kappa-\kappa')L} - a(\zeta)^2\,\mathrm{e}^{\mathrm{i}2(\kappa'-\kappa)L}]. \tag{29}$$

Since $|\zeta|, |\zeta'| > \rho$, and $2(\kappa - \kappa') = (\zeta - \zeta')(1 + \rho^2\zeta^{-1}\zeta'^{-1})$, we have

$$\lim_{L\to\infty} \frac{1}{\mathrm{i}(\zeta - \zeta')}\,\mathrm{e}^{\mathrm{i}2(\kappa-\kappa')L} = \pi\delta(\zeta - \zeta'). \tag{30}$$

Then taking $L \to \infty$, (28) gives

$$\langle\Phi(\zeta')|\Psi(\zeta)\rangle = 2\pi a(\zeta)^2(1 - \rho^2\zeta^{-2})\delta(\zeta - \zeta'), \tag{31}$$

where

$$\langle\Phi(\zeta')|\Psi(\zeta)\rangle = \lim_{L\to\infty}\int_{-L}^{L} \mathrm{d}x(f_1g_1 + f_2g_2). \tag{32}$$

Similarly, we have

$$\langle\tilde{\phi}(\zeta')|\tilde{\Psi}(\zeta)\rangle = -2\pi\tilde{a}(\zeta)^2(1 - \rho^2\zeta^{-2})\delta(\zeta - \zeta'). \tag{33}$$

6. Choice of ζ in the whole range $\{-\infty, \infty\}$

It is well known that

$$\tilde{\psi}(x,\zeta) = \mathrm{i}\rho^{-1}\eta\psi(x,\eta), \qquad \tilde{\phi}(x,\zeta) = -\mathrm{i}\rho^{-1}\eta\phi(x,\eta), \qquad \eta = \rho^2\zeta^{-1}, \tag{34}$$

which means that $\tilde{\psi}(x,\zeta)$ with $|\zeta| > \rho$ is proportional to $\psi(x,\eta)$ with $|\eta| = \rho^2|\zeta|^{-1} < \rho$.

We now show that (33) can be written in the form of (31). From (34) we have

$$\tilde{\Psi}(x,\zeta) = -\rho^{-2}\eta^2\Psi(x,\eta), \qquad \tilde{\Phi}^A(x,\zeta') = -\rho^{-2}\eta'^2\Psi(x,\eta'), \tag{35}$$

where $\zeta = \rho^2\eta^{-1}$, $\zeta' = \rho^2\eta'^{-1}$ and then

$$\delta(\zeta - \zeta') = \delta(\rho^2\eta^{-1} - \rho^2\eta'^{-1}) = \frac{\eta\eta'}{\rho^2}\delta(\eta - \eta'). \tag{36}$$

Therefore, substituting (35) and (36) into (33), we get

$$\langle\Phi(\eta')|\Psi(\eta)\rangle\rho^{-4}\eta'^2\eta^2 = -2\pi a(\eta)^2(1 - \rho^{-2}\eta^2)\frac{\eta\eta'}{\rho^2}\delta(\eta - \eta'). \tag{37}$$

Then by eliminating the same factors on two sides, we obtain

$$\langle\Phi(\eta')|\Psi(\eta)\rangle = 2\pi a(\eta)^2(1 - \rho^2\eta^{-2})\delta(\eta - \eta'), \tag{38}$$

which has the same form as (31), except $|\eta|, |\eta'| < \rho$. This means that (33) with $|\zeta|, |\zeta'| > \rho$ is equivalent to (31) with $|\zeta|, |\zeta'| < \rho$. Hence, ζ, ζ' in (31) can be extended to the whole domain $\{-\infty, \infty\}$ to let (31) contain (33). Then the orthogonality relations reduce to only one, i.e. (31) with $-\infty < \zeta, \zeta' < \infty$. Accordingly, the unique independent Jost solution is

$\psi(x, \zeta)$; the independent squared Jost solution and adjoint squared solution are $\Psi(x, \zeta)$ and $\Phi^A(x, \zeta)$, respectively.

After extending the argument ζ to the whole real axis, the squared Jost function $\Psi(x, \zeta)$ can be analytically continued to the upper half ζ-plane. Then from

$$\mathcal{L}(u)\Psi(x, t, \zeta) = 0, \qquad \Psi(x, t, \zeta) = \begin{pmatrix} \psi_1^2(x, t, \zeta) \\ \psi_2^2(x, t, \zeta) \end{pmatrix}, \tag{39}$$

we have

$$\mathcal{L}(u)\Psi(x, t, \zeta_n) = 0, \tag{40}$$

$$\mathcal{L}(u)\dot{\Psi}(x, t, \zeta_n) = 0, \qquad \dot{\Psi}(x, t, \zeta_n) = \frac{\mathrm{d}}{\mathrm{d}\zeta}\Psi(x, t, \zeta)\bigg|_{\zeta=\zeta_n}, \tag{41}$$

where $\mathcal{L}(u)$ does not contain the parameter ζ, and ζ_n is one of the zeros of $a(\zeta)$.

Applying operators $\frac{\mathrm{d}}{\mathrm{d}\zeta}\big|_{\zeta=\zeta'=\zeta_n}$, $\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\big|_{\zeta=\zeta'=\zeta_n}$ and $\left(\frac{\mathrm{d}^3}{\mathrm{d}\zeta^3} + 3\frac{\mathrm{d}}{\mathrm{d}\zeta'}\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\right)\big|_{\zeta=\zeta'=\zeta_n}$ to (29) respectively, we obtain

$$\begin{aligned} &\langle\Phi(\zeta_m)|\Psi(\zeta_n)\rangle = 0, \\ &\langle\dot{\Phi}(\zeta_m)|\Psi(\zeta_n)\rangle = \langle\Phi(\zeta_m)|\dot{\Psi}(\zeta_n)\rangle = \mathrm{i}\dot{a}(\zeta_n)^2(1 - \rho^2\zeta_n^{-2})\delta_{mn}, \\ &\langle\dot{\Phi}(\zeta_m)|\dot{\Psi}(\zeta_n)\rangle = \mathrm{i}\dot{a}(\zeta_n)\ddot{a}(\zeta_n)(1 - \rho^2\zeta_n^{-2})\delta_{mn} + \mathrm{i}2\rho^2\zeta_n^{-3}\dot{a}(\zeta_n)^2\delta_{mn}. \end{aligned} \tag{42}$$

7. Green's function

The linearized equation is essentially a linear equation, and we can choose the appropriate method for it similar to that for linear equations. The rhs of linearized equation (5) is not zero, then its Green function $G(x, t; x', t')$ can be defined as

$$\mathcal{L}(u)G = \delta(x - x')\delta(t - t'), \tag{43}$$

that is a 2×2 matrix since the basic solution has two components. The solution of (5) can be written as

$$\boldsymbol{q} = \mathrm{i}\int_0^\infty \mathrm{d}t' \int_{-\infty}^\infty \mathrm{d}x' G(x, t; x', t')\boldsymbol{P}(x', t'). \tag{44}$$

Because of the causality condition, G should be zero for $t' > t$; hence

$$G = G^0(x, t; x', t')\theta(t - t'), \tag{45}$$

where $\theta(\tau)$ represents the step function, i.e. $\theta(\tau) = 1$ for $\tau > 0$ and $\theta(\tau) = 0$ for $\tau < 0$. Then (44) becomes

$$\boldsymbol{q} = \mathrm{i}\int_0^t \mathrm{d}t' \int_{-\infty}^\infty \mathrm{d}x' G^0(x, t; x', t')\boldsymbol{P}(x', t'), \tag{46}$$

this solution fulfils the initial condition $q = 0$ for $t = 0$. Substituting (46) into (5), we have

$$\int_{-\infty}^\infty \mathrm{d}x' G^0(x, t; x', t)\boldsymbol{P}(x', t) + \int_0^t \mathrm{d}t' \int_{-\infty}^\infty \mathrm{d}x' \mathcal{L}(u)G^0(x, t; x', t')\boldsymbol{P}(x', t') = \boldsymbol{P}(x, t). \tag{47}$$

This equation is identically satisfied by assuming that G^0 is a solution of the homogeneous equation

$$\mathcal{L}(u)G^0(x, t; x', t') = 0, \tag{48}$$

and obeys the final condition

$$G^0(x,t;x',t')|_{t=t'} = \delta(x-x'), \tag{49}$$

noting that these are all 2×2 matrices. The Green function G^0 is completely determined by the above two conditions.

8. Proof of the completeness

As we have seen in (39)–(41), the squared Jost function $\Psi(x,t,\zeta)$, $\Psi(x,t,\zeta_n)$ and $\dot{\Psi}(x,t,\zeta_n)$ are the basic solutions of the linearized equation, i.e. solutions of the homogeneous equation. Moreover, from the properties of the reduction transformation, the state labelled with a tilde is not needed. Therefore,

$$\begin{aligned} G^0(x,t;x',t') = \int_\Gamma \mathrm{d}\zeta\, \Psi(x,t;\zeta)A(x',t';\zeta) + \sum_n \Psi(x,t;\zeta_n)B(x',t';\zeta_n) \\ + \sum_n \dot{\Psi}(x,t;\zeta_n)C(x',t';\zeta_n), \end{aligned} \tag{50}$$

where A, B, C are undetermined 1×2 matrices, and the integral path Γ is along the real axis from $-\infty$ to ∞, but goes over the origin in the upper ζ-plane. Then (49) becomes

$$\begin{aligned} \delta(x-x') = \int_\Gamma \mathrm{d}\zeta\, \Psi(x,t;\zeta)A(x',t;\zeta) + \sum_n \Psi(x,t;\zeta_n)B(x',t;\zeta_n) \\ + \sum_n \dot{\Psi}(x,t;\zeta_n)C(x',t;\zeta_n). \end{aligned} \tag{51}$$

Multiplying by $\Phi^A(x,t;\zeta')^T$ from the left and integrating over x, (51) yields

$$\Phi^A(x',t;\zeta')^T = \int_\Gamma \mathrm{d}\zeta\, M(t;\zeta',\zeta)A(x',t;\zeta), \tag{52}$$

where

$$M(t;\zeta',\zeta) = \int_{-\infty}^{\infty} \mathrm{d}x\, \Phi^A(x,t;\zeta)^T\Psi(x,t;\zeta) = 2\pi a(\zeta)^2(1-\rho^2\zeta^{-2})\delta(\zeta-\zeta'), \tag{53}$$

the summary terms vanish for the orthogonality (42), and the final expression of M comes from (31). Then from (52) and (53), we have

$$A(x',t;\zeta) = \frac{1}{2\pi a(\zeta)^2(1-\rho^2\zeta^{-2})}\Phi^A(x',t;\zeta)^T. \tag{54}$$

Similarly, multiplying (51) respectively by $\Phi^A(x,t;\zeta_m)^T$ and $\dot{\Phi}^A(x,t;\zeta_m)^T$ from the left and integrating over x yields

$$\Phi^A(x',t;\zeta_m) = \sum_n N(t;\zeta_m,\zeta_n)C(x',t;\zeta_n), \tag{55}$$

$$\dot{\Phi}^A(x',t;\zeta_m) = \sum_n N(t;\zeta_m,\zeta_n)B(x',t;\zeta_n) + \sum_n Q(t;\zeta_m,\zeta_n)C(x',t;\zeta_n), \tag{56}$$

where

$$N(t;\zeta_m,\zeta_n) = \int_{-\infty}^{\infty} \mathrm{d}x\, \Phi^A(x,t;\zeta_m)^T\dot{\Psi}(x,t;\zeta_n) = \mathrm{i}\dot{a}(\zeta_n)^2\left(1-\rho^2\zeta_n^{-2}\right)\delta_{mn}, \tag{57}$$

and

$$Q(t;\zeta_m,\zeta_n) = \int_{-\infty}^{\infty} \mathrm{d}x\, \dot{\Phi}^A(x,t;\zeta_m)^T \dot{\Psi}(x,t;\zeta_n)$$
$$= \mathrm{i}\dot{a}(\zeta_n)\ddot{a}(\zeta_n)\big(1-\rho^2\zeta_n^{-2}\big)\delta_{mn} + \mathrm{i}2\dot{a}(\zeta_n)^2\rho^2\zeta_n^{-3}\delta_{mn}. \tag{58}$$

From the above four equations, we have

$$C(x',t;\zeta_n) = -\mathrm{i}\frac{1}{\dot{a}(\zeta_n)^2\big(1-\rho^2\zeta_n^{-2}\big)}\Phi^A(x',t;\zeta_n)^T, \tag{59}$$

$$B(x',t;\zeta_n) = \mathrm{i}\left(\frac{\ddot{a}(\zeta_n)}{\dot{a}(\zeta_n)^3(1-\rho^2\zeta_n^{-2})} + \frac{2\rho^2\zeta_n^{-3}}{\dot{a}(\zeta_n)^2\big(1-\rho^2\zeta_n^{-2}\big)^2}\right)\Phi^A(x',t;\zeta_n)^T. \tag{60}$$

Substituting the expressions of A, B and C into (47), we obtain the equation of completeness,

$$\delta(x-x') = \frac{1}{2\pi}\int_\Gamma \mathrm{d}\zeta \frac{1}{a(\zeta)^2(1-\rho^2\zeta^{-2})}\Psi(x,\zeta)\Phi^A(x',\zeta)^T$$
$$-\mathrm{i}\sum_{n=1}^{N}\frac{1}{\dot{a}(\zeta_n)^2\big(1-\rho^2\zeta_n^{-2}\big)}\big(\dot{\Psi}(x,\zeta_n)\Phi^A(x',\zeta_n)^T + \Psi(x,\zeta_n)\dot{\Phi}^A(x',\zeta_n)^T\big)$$
$$+\mathrm{i}\sum_{n=1}^{N}\left(\frac{\ddot{a}(\zeta_n)}{\dot{a}(\zeta_n)^3\big(1-\rho^2\zeta_n^{-2}\big)} + \frac{2\rho^2\zeta_n^{-3}}{\dot{a}(\zeta_n)^2\big(1-\rho^2\zeta_n^{-2}\big)^2}\right)\Psi(x,\zeta_n)\Phi^A(x',\zeta_n)^T. \tag{61}$$

Thus, the completeness relation is obtained; it is the same as the one in [13] which are proved by the generalized Marchenko equation.

9. Discussion

The NLS$^+$ equation has dark-soliton solutions under the condition of non-vanishing boundary ($u \to \rho$ as $x \to \infty$ and $u \to \rho e^{\mathrm{i}\alpha}$ as $x = -\infty$) [25]. Since the perturbation theory based upon IST requires fixed-boundary condition, it cannot be applied to nonlinear equations with non-vanishing boundary which might be changed by corrections. As a result, the direct perturbation theory is a more appropriate choice for this kind of problem. Pioneering work on this direction is due to Konotop and Vekslerchik [11]. Unfortunately, their choice of the basic solutions for the linearized equation did not consist of independent solutions, which resulted in contradiction in determining the adiabatic variation of the parameters characterizing solutions. To avoid this problem, they assumed that the boundary is not determined by corrections but unvaried or only varied in a given manner. With the explicit expression of the single-dark-soliton solution of NLS$^+$ equation, the independent basic solutions were correctly chosen later in [12], and then the orthogonality and completeness of the basic solutions were shown obviously. The method was soon extended to the multi-dark-soliton case [13], in which the orthogonal relations were derived by Wronskian determinant and the complete relations were proved by the generalized Marchenko equation. Recently, the authors of [14, 15] re-examined this problem, but they contain obvious mistakes that we should clarify as follows.

In the light-soliton case of NLS equation, only two of the two-component Jost functions are independent because the compatible operator is a 2×2 matrix of first-order derivatives and the other two depend on them through the monodromy matrix. But in the dark-soliton case of NLS$^+$ equation, two values of the affine parameter ζ correspond to only a single λ in the compatible pair. Therefore, we take $|\zeta| > \rho$ in order to guarantee one-to-one mapping. Within this restriction, the independent two-component solutions can be chosen as $\tilde{\psi}(x,t,\zeta)$ and $\psi(x,t,\zeta)$. In the following theory, the range of ζ should be extended to the whole axis and then continued to the complex plane. However, after we take $-\infty < \zeta < \infty$, there remains only one independent two-component solution, either $\tilde{\psi}(x,t,\zeta)$ or $\psi(x,t,\zeta)$. Detailed consideration of these choices of the range of ζ has been given in sections 4 and 6. The paper of Konotop and Vekslerchik [11] is a pioneering work in this direction. Unfortunately, they chose two independent two-component solutions $\Psi(x,\zeta)$ and $\tilde{\Psi}(x,\zeta)$ in the whole domain of $-\infty < \zeta < \infty$. Recently in [14], the authors take $\Psi(x,\zeta)$ and $\Phi(x,\zeta)$ as two independent solutions of the linearized operator in whole ζ domain at the same time. Such a selection obviously results in a serious contradiction, because the Jost solutions $\phi(x,\zeta)$ and $\tilde{\psi}(x,\zeta)$ are proportional to each other in the case of reflectionlessness, i.e. $\phi(x,\zeta) = a(\zeta)\tilde{\psi}(x,\zeta)$; the eigenfunction $\Phi(x,\zeta)$ is just $\tilde{\Psi}(x,\zeta)$ except the factor $a(\zeta)^2$. Such mistake is almost the same as that in [11].

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